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Passive scalar turbulence in high dimensions

Andrea Mazzino[1] and Paolo Muratore-Ginanneschi[2]
[1]*INFM-Department of Physics, University of Genova, I-16146 Genova, Italy*
[2]*The Niels Bohr Institute, Blegdamsvej 17, DK-2100 Copenhagen, Denmark*
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Exploiting a Lagrangian strategy we present a numerical study for both perturbative and nonperturbative regions of the Kraichnan advection model. The major result is the numerical assessment of the first-order $1/d$ expansion by Chertkov, Falkovich, Kolokolov, and Lebedev [Phys. Rev. E **52**, 4924 (1995)] for the fourth-order scalar structure function in the limit of high dimension d's. In addition to the perturbative results, the behavior of the anomaly for the sixth-order structure functions versus the velocity scaling exponent, ξ, is investigated and the resulting behavior is discussed.

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A first principles theory of turbulent advection was very recently developed in the framework of the passive scalar model introduced by Kraichnan [1,2] where the velocity field advecting the scalar is Gaussian in space and white in time (i.e., δ-correlated). The main feature of such model is that its statistical description can be obtained from the solution of a set of closed linear partial differential equations for the equal-time correlation functions [1].

It was conjectured by Kraichnan [2] that, despite the closure of the equations for the equal-time correlation functions, the system should display an *intermittent* behavior. This means [3] that the scaling exponents ζ_{2n} of the $2n$-order scalar structure functions S_{2n} are *anomalous*, namely, $S_{2n} \equiv \langle [\theta(\mathbf{r},t)-\theta(\mathbf{0},t)]^{2n}\rangle \propto r^{\zeta_{2n}}$ with $\zeta_{2n} < n\zeta_2$ as r goes to zero. This is in sharp contrast with the *normal* scaling $\zeta_{2n} = n\zeta_2$ predicted on the basis of mean-field dimensional arguments.

Intermittent behavior was subsequently established by Gawędzki and Kupiainen [4], Chertkov *et al.* [5], and Shraiman and Siggia [6,7]. Anomalous scaling of the structure functions appears because of the presence of zero modes of the equations for the correlation functions [4–9].

The second-order structure function was solved exactly in Ref. [2] and it does not show any anomaly. Corrections to normal scaling start to appear for the fourth-order structure functions (the third, when scalar fluctuations are sustained by a large-scale gradient [10]). The anomalous exponents have been up to now computed only perturbatively around limit cases where the structure functions of the Kraichnan model are known and display normal scaling. The corresponding perturbative regions are: small ξ's [4,11], large d's [5,9], and ξ close to the Batchelor limit $\xi=2$ [6,7,12,13], ξ being the scaling exponent of the advecting velocity field and d the dimension of the space. The first two expansions are regular, while for the third one the relevant small parameter should be $\sqrt{2-\xi}$. This is due to the preservation of the collinear geometry in the Batchelor limit, leading to an angular nonuniformity in the perturbation analysis [14].

The scaling behavior of high order structure functions was investigated by means of instanton calculus [15]. The solutions obtained up to now predict the saturation to a constant value of the scaling exponents ζ_n's as n becomes large [16–18].

It is of great interest both to develop efficient numerical strategies to test the predictions of the available perturbation theories (and thus to give numerical confirmations of the mechanisms responsible for the emergence of intermittency) and to investigate the nonperturbative regions, still not completely accessible by analytical approaches. Exploiting a new numerical strategy such program was pursued in Refs. [19–21] (see also the review paper [22]) where, in particular, the behavior of the anomaly $\sigma_{2n} \equiv n\zeta_2 - \zeta_{2n}$ was studied as a function of ξ for $n=2$ both in two and three dimensions. The numerical experiments confirmed the perturbative predictions for $\xi \to 0$ and $\xi \to 2$. However, due to the difficulty of the numerical simulations no results were available for the behavior of σ_{2n} as a function of ξ for moments higher than the fourth and far from the two above perturbative regions. A numerical study of the curve σ_6 versus ξ in the nonperturbative region of the Kraichnan model is one of the aims of the present Rapid Communication.

The second aim is the investigation of the Kraichnan model in large spatial dimensions. The $1/d$ expansion received solely a partial numerical confirmation for the third-order structure function [23]. The value of d investigated in Ref. [23] was not large enough to reach the perturbative regime for the fourth-order structure function and thus to verify the perturbative prediction of Ref. [5]. This is a second goal of the present paper. Exploiting the Lagrangian strategy presented in Refs. [19,20] we present two sets of numerical simulations which make possible the numerical assessment of the $1/d$ expansion of Ref. [5]. We found that, for $\xi=0.8$, the value of d at which the perturbative regime studied in Ref. [5] takes place is $d \sim 30$. The latter value reduces as ξ decreases, being of the order of 25 for $\xi=0.6$.

Let us briefly recall the Kraichnan advection model [1,2]. In this model the velocity field $\mathbf{v}=\{v_\alpha, \alpha=1,\ldots,d\}$ advecting the scalar is incompressible, isotropic, Gaussian in space, white-noise in time; it has homogeneous increments with power-law spatial correlations and a scaling exponent ξ in the range $0<\xi<2$:

$$\langle [v_\alpha(\mathbf{r},t)-v_\alpha(\mathbf{0},0)][v_\beta(\mathbf{r},t)-v_\beta(\mathbf{0},0)]\rangle = 2\delta(t)D_{\alpha\beta}(\mathbf{r}), \tag{1}$$

where

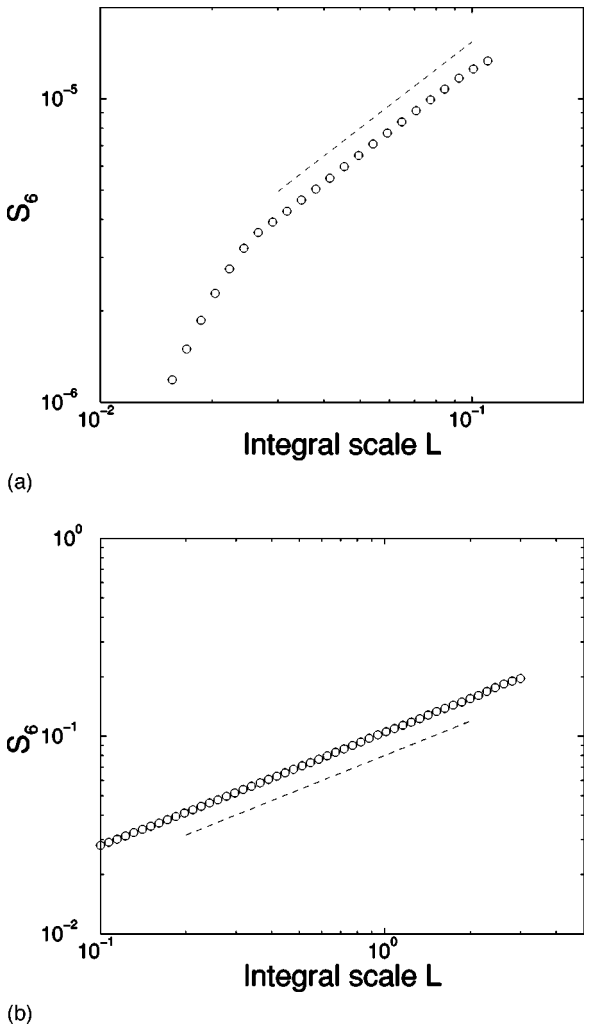

FIG. 1. Three-dimensional sixth-order structure functions S_6 versus L for $\xi=0.9$ (top) and $\xi=1.5$ (bottom). The dashed straight lines are the best fit slopes calculated in the scaling region. Their values give the anomalies reported in Fig. 2 (upper curve). Number of realizations $\sim 10^7$.

$$D_{\alpha\beta}(\mathbf{r}) = D_0 r^{\xi}\left[(\xi+d-1)\,\delta_{\alpha\beta} - \xi\frac{r_\alpha r_\beta}{r^2}\right]. \tag{2}$$

The specific form inside the squared brackets is dictated by the incompressibility condition on $\mathbf{v}$.

The nonintermittent velocity field defined by Eq. (1) advects a scalar field θ

$$\partial_t\theta + \mathbf{v}\cdot\boldsymbol{\partial}\theta = \kappa\,\partial^2\theta + f \tag{3}$$

of which we want to investigate the statistical properties related to intermittency. Here f is an external forcing and κ is the molecular diffusivity.

The forcing term permits us to attain a stationary state defined by the balance between production of scalar variance (related to f) and its dissipation (related to $\kappa\partial^2\theta$). The net result is that $\langle\theta^2\rangle$ is finite in the steady state.

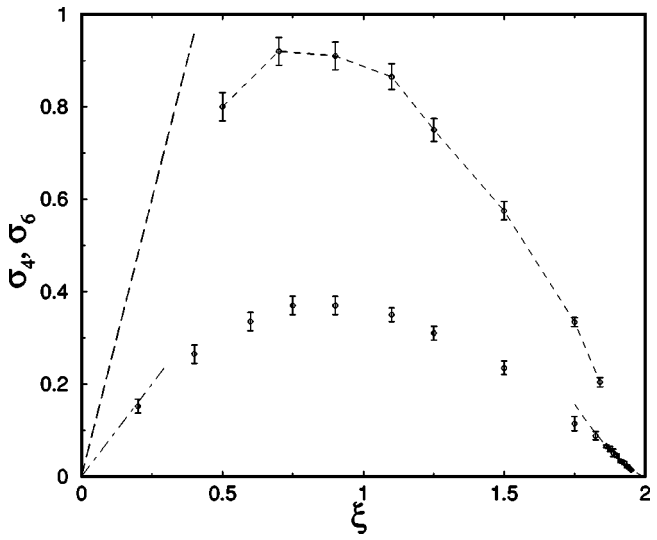

FIG. 2. Anomaly $\sigma_6 \equiv 3\zeta_2 - \zeta_6$ for the sixth-order structure functions in three dimensions (upper curve) and, for comparison, the σ_4 curve in Ref. [19] (lower curve). The dashed straight line on the left is the first-order prediction $\sigma_6 = 12\,\xi/5$ by Bernard *et al.* [8]. Notice the shift on the left of the maximum of the σ_6 curve with respect to the maximum of σ_4.

We shall assume the (random) forcing f of zero mean, isotropic, Gaussian, white-noise in time and homogeneous. Its correlation is specified by:

$$\langle f(\mathbf{r},t)\, f(\mathbf{0},0)\rangle = \chi(r/L)\,\delta(t), \tag{4}$$

with $\chi(0)>0$ and $\chi(r/L)$ a rapidly decreasing function for $r \gg L$ where L is the (forcing) integral scale.

Rather than integrating the partial stochastic differential equation (3) (that is a prohibitive task already in two dimensions due to the δ correlation of $\mathbf{v}$), we exploit the Lagrangian formulation of the passive scalar dynamics as in Refs. [19,20] in order to simulate particle trajectories by Monte Carlo methods.

We recall that when such method is adopted the evaluation of structure functions is reduced to the study of the statistical properties of the random variable describing the time spent by pairs of particle with mutual distance less than the integral scale L. Generally, the distance between pairs of particles tends to increase with the elapsed time but, occasionally, particles may come very close and stay so; the phenomenon is the source of scaling anomalies. The main advantage of the Lagrangian strategy is that one does not have to generate the whole velocity field, but just the velocity field over the Lagrangian trajectories. Simulations in very high space dimensions as well as the investigation of high-order moments become thus feasible.

We now present results for structure functions up to sixth order. The three-dimensional results relative to the fourth-order structure functions have been already published in Ref. [19]. They will be reported in the following for comparison with our new data.

The L dependence of $S_6(r;L)$ is shown for the three-dimensional case in Fig. 1 for $\xi=0.9$ (top) and $\xi=1.5$ (bot-

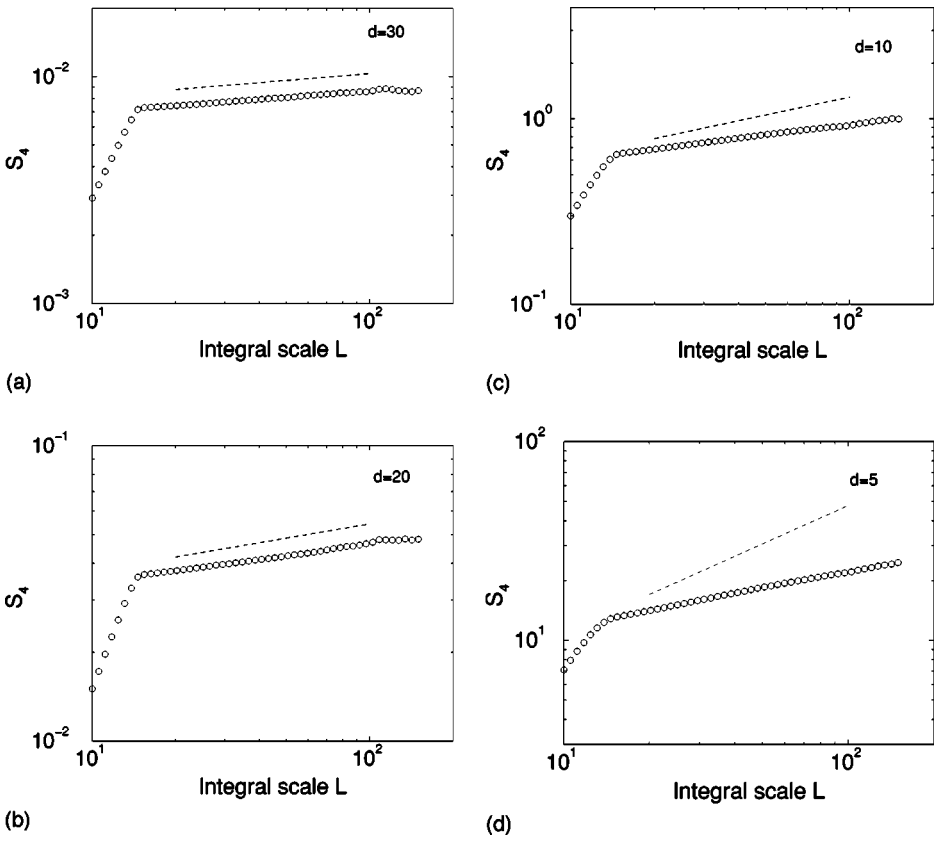

FIG. 3. Fourth-order structure functions for $\xi=0.8$ and d ranging from 5 to 30. Dashed straight lines represent the anomaly from the $1/d$ expansion by Chertkov *et al.* [5]. Number of realizations $\sim 10^6$.

tom). Similar plots have been obtained also for other values of ξ. In both cases the scaling region is indicated by a dashed straight line, the slope of which yields the anomaly $\sigma_6 \equiv 3\zeta_2 - \zeta_6$. The measured slopes for $\xi=0.9$ and $\xi=1.5$ are $\sigma_6 = 0.91 \pm 0.03$ and $\sigma_6 = 0.58 \pm 0.02$, respectively. The error bars are obtained by analyzing the fluctuations of local scaling exponents over smaller ratios of values for L. As explained in Ref. [20] the number of realizations needed to obtain high quality scaling as the ones displayed in [19] increases rapidly with decreasing ξ. This is the reason why we restricted our analysis to the interval $\xi \geqslant 0.5$.

The results for the anomalies versus ξ are summarized in Fig. 2. The upper curve refers to σ_6, and the lower curve to σ_4 (the latter has been already published in Ref. [19]).

Some remarks on the behavior of σ_6 are worth mentioning. First, we note that when ξ decreases from 2 to 0 the anomaly σ_6 grows at first, achieves a maximum, and finally decreases as in the case of σ_4.

We have evidence that σ_6 tends to vanish for $\xi \to 0$ as follows from the perturbative predictions [11,8] the leading order of which $\sigma_6 = 12\,\xi/5$ is shown as a dashed straight line on the left-hand side of Fig. 2. The fact that the anomaly for S_6 is higher than the one for S_4 is an immediate consequence of Hölder inequalities [3].

More interesting is the fact that the maximum of the anomaly occurs for a value of ξ which for S_6 is smaller than for S_4. This can be explained as follows. Near $\xi=0$ the dynamics is dominated by the nearly ultraviolet-divergent eddy diffusion. The latter arises from small-scale fluctuations averaged over time scales much larger than the typical ones. Large fluctuations are thus averaged out and appear strongly

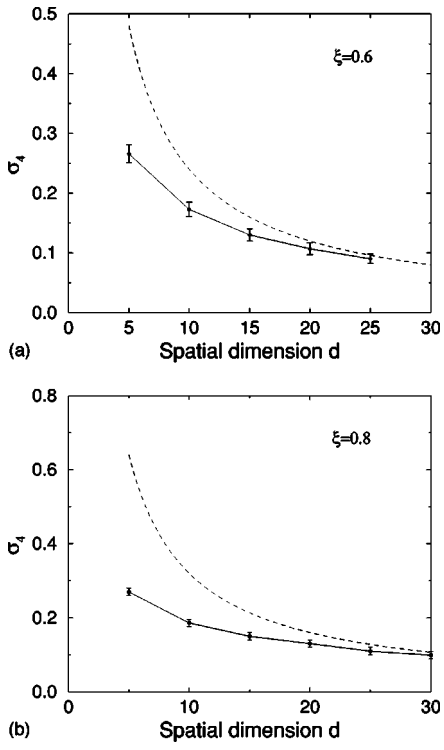

FIG. 4. Anomalies σ_4 versus d for $\xi=0.6$ and $\xi=0.8$. Dashed lines are the anomalies $4\xi/d$ from the $O(1/d)$ perturbation theory of Ref. [5].

depleted. On the contrary, near $\xi=2$ the dynamics is dominated by the nearly infrared-divergent stretching. No averaging of the large fluctuations now takes place and the net result is that they strongly contribute to the scalar statistics. Since higher moments are more sensitive to large fluctuations than the lower ones, one expects stretching to dominate the dynamics for wider ranges of ξ as the order n of the moment increases. As a consequence the value of ξ where the effects of stretching and diffusion balance (i.e., where the maximum of the anomaly is expected) should move toward the left as n increases in agreement with the results reported in Fig. 2. Let us now turn to the behavior of fourth-order structure functions for large space dimensions. In such limit the first-order perturbative prediction in $1/d$ gives for the fourth-order structure functions the anomaly $4\xi/d$ [5,9]. The Lagrangian strategy allows us to verify numerically the perturbative prediction.

We performed two sets of simulations for $\xi=0.6$ and $\xi=0.8$ and different spatial dimensions, d, from 5 to 30. The fourth-order structure functions for $d=5$, 10, 20, and 30 are shown in Fig. 3 for $\xi=0.8$. Similar scaling laws have been observed also for $\xi=0.6$. Dashed straight lines represent the slopes $4\xi/d$. For $d=30$ the anomaly obtained from numerical simulations (i.e., the slopes of the curve with circles) is practically indistinguishable from the perturbative expression. The discrepancy increases rapidly as d reduces. The results are summarized in Fig. 4, where the anomaly σ_4 is shown as a function of d for $\xi=0.6$ and $\xi=0.8$. The fact that for $\xi=0.6$ the perturbative regime starts for a value of d (~ 25) lower than the one at $\xi=0.8$ ($d\sim 30$) is the consequence of the fact that the small parameter in the perturbation theory is $\propto 1/[d(2-\xi)]$ rather than $1/d$.

In conclusion, we verified the $1/d$ expansion for the fourth-order structure functions. We presented two sets of simulations corresponding to two different values of the velocity scaling exponent ξ. For $\xi=0.8$ the perturbative regime sets in for $d\sim 30$. This value reduces at $d\sim 25$ for $\xi=0.6$. The result is expected as $1/[d(2-\xi)]$ is the relevant small parameter in the perturbation theory. We also studied the behavior of the anomaly for the sixth-order structure functions versus the velocity scaling exponent ξ. We identified and discussed two competing mechanisms which control the position of the maximum of the anomaly along the ξ axis.

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